RIMS Kôkyûroku Bessatsu
B59 (2016), 203–214

Stochastic differential equations related to random matrix theory

By

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Abstract

In this note we review recent results on existence and uniqueness of solutions of infinite-dimensional stochastic differential equations describing interacting Brownian motions on $\mathbb{R}^d$.

§ 1. Introduction

Let $\mathbf{X}^N(t) = (X_j^N(t))_{j=1}^N$ be a solution of the stochastic differential equation (SDE)

$$dX_j^N(t) = dB_j(t) + \frac{\beta}{2} \sum_{k=1, k\neq j}^{N} \frac{dt}{X_j^N(t) - X_k^N(t)} \tag{1.1}$$

or the SDE with Ornstein-Uhlenbeck's type drifts

$$dX_j^N(t) = dB_j(t) - \frac{\beta}{4N} X_j^N(t)dt + \frac{\beta}{2} \sum_{k=1, k\neq j}^{N} \frac{dt}{X_j^N(t) - X_k^N(t)}, \tag{1.2}$$

where $B_j(t), j = 1, 2, \ldots, N$ are independent one-dimensional Brownian motions. These are called Dyson's Brownian motion models with parameters $\beta > 0$ [4]. They were introduced to understand the statistics of eigenvalues of random matrix ensembles as

Received January 31, 2016. Revised May 14, 2016.
2010 Mathematics Subject Classification(s): 15B52, 30C15, 47D07, 60G55, 82C22
Key Words: *Random matrices, Interacting particle systems, Stochastic differential equations.*
Supported by a Grant-in-Aid for Scientific Research (KIBAN-A, No.24244010) and a Grant-in-Aid for Scientific Research (KIBAN-C, No.15K04916)
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distributions of particle positions in one-dimensional Coulomb gas systems with log-potential.

The solution of (1.2) is a natural reversible stochastic dynamics with respect to $\check{\mu}^N_{\mathsf{bulk},\beta}$:

$$\check{\mu}^N_{\mathsf{bulk},\beta}(d\mathbf{x}_N) = \frac{1}{Z} h_N(\mathbf{x}_N)^\beta e^{-\frac{\beta}{4N}|\mathbf{x}_N|^2} d\mathbf{x}_N, \tag{1.3}$$

where $d\mathbf{x}_N = dx_1 dx_2 \cdots dx_N$, $\mathbf{x}_N = (x_i) \in \mathbb{R}^N$, and

$$h_N(\mathbf{x}_N) = \prod_{i<j}^N |x_i - x_j|.$$

Throughout, Z denotes a normalizing constant. Gaussian ensembles are called Gaussian orthogonal/unitary/symplectic ensembles (GOE/GUE/GSE) according to their invariance under conjugation by orthogonal/unitary/symplectic groups, which correspond to the inverse temperatures $\beta = 1, 2$ and 4, respectively [9, 2]. It is natural to believe that the N-limit of the process $\mathbf{X}^N(t)$ solves the infinite-dimensional stochastic differential equation (ISDE)

$$dX_j(t) = dB_j(t) + \frac{\beta}{2} \lim_{r\to\infty} \sum_{\substack{k=1,k\neq j \\ |X_k(t)|<r}}^{\infty} \frac{dt}{X_j(t) - X_k(t)}. \tag{1.4}$$

The result was not proved rigorously until a few years ago when it was shown for $\beta = 2$ in [17], for $\beta = 1, 2, 4$ in [8], and for $\beta \geq 1$ in [23] .

Set $Y_j^N(t) = N^{1/6}(X_j^N(t) - 2\sqrt{N})$, $j = 1, 2, \ldots . N$ for the solution $\mathbf{X}^N$ of (1.2). It has also been shown that the N-limit of the process $\mathbf{Y}^N(t)$ solves the ISDE

$$dY_j(t) = dB_j(t) + \frac{\beta}{2} \lim_{r\to\infty} \left\{ \sum_{\substack{k=1,k\neq j \\ |Y_k(t)|<r}}^{\infty} \frac{1}{Y_j(t) - Y_k(t)} - \int_{-r}^{r} \frac{\widehat{\rho}(x)dx}{-x} \right\} dt, \tag{1.5}$$

with $\widehat{\rho}(x) = \pi^{-1}\sqrt{-x}\mathbf{1}(x < 0)$, for $\beta = 2$ [17] and for $\beta = 1, 2, 4$ [8].

One of the key parts of proving the above results is the existence and uniqueness of solutions of an ISDE of the form

$$dX_j(t) = dB_j(t) - \frac{1}{2}\nabla\Phi(X_j(t))dt - \frac{1}{2} \sum_{k=1,k\neq j}^{\infty} \nabla\Psi(X_j(t), X_k(t))dt \tag{1.6}$$

with free potential Φ and interaction (pair) potential Ψ. In ISDEs (1.4) and (1.5), Ψ is given by the log pair potential $-\beta \log |x - y|$. The present note is a short summary of results on existence and uniqueness of solutions for ISDE (1.6).

§ 2. Quasi-Gibbs measure

Let S be a closed set in $\mathbb{R}^d$ such that $0 \in S$ and $\overline{S^{\mathrm{int}}} = S$, where S^{int} denotes the interior of S. The configuration space $\mathfrak{M}$ of unlabelled particles is given by

$$
\begin{aligned}
(2.1)\qquad \mathfrak{M} &= \Big\{\xi : \xi \text{ is a nonnegative integer valued Radon measure in } S\Big\} \\
&= \Big\{\xi(\cdot) = \sum_{j\in\mathbb{I}} \delta_{x_j}(\cdot) : \sharp\{j \in \mathbb{I} : x_j \in K\} < \infty, \text{ for any } K \text{ compact}\Big\},
\end{aligned}
$$

where $\mathbb{I}$ is a countable set and δ_a is the Dirac measure at $a \in S$. Thus $\mathfrak{M}$ is a Polish space with the vague topology. We also introduce a subset $\mathfrak{M}_{\mathrm{s.i}}$ of $\mathfrak{M}$:

$$
(2.2)\qquad \mathfrak{M}_{\mathrm{s.i.}} = \{\xi \in \mathfrak{M} : \xi(\{x\}) \le 1 \text{ for all } x \in S,\ \xi(S) = \infty\},
$$

that is, the set of configurations of an infinite number of particles without collisions. For Borel measurable functions $\Phi : S \to \mathbb{R} \cup \{\infty\}$ and $\Psi : S \times S \to \mathbb{R} \cup \{\infty\}$ and a given increasing sequence $\{b_r\}$ of $\mathbb{N}$, we introduce the Hamiltonian

$$
(2.3)\qquad H_r(\xi) = H_r^{\Phi,\Psi}(\xi) = \sum_{x_j \in S_r} \Phi(x_j) + \sum_{x_j, x_k \in S_r, j<k} \Psi(x_j, x_k), \quad \xi = \sum_{j\in\mathbb{I}} \delta_{x_j},
$$

where $S_r = \{x \in S : |x| < b_r\}$. We call Φ a free potential, and call Ψ an interaction potential. Let Λ_r^m be the restriction of a Poisson random measure with intensity measure dx on $\mathfrak{M}_r^m = \{\xi \in \mathfrak{M} : \xi(S_r) = m\}$. We define maps $\pi_r, \pi_r^c : \mathfrak{M} \to \mathfrak{M}$ such that $\pi_r(\xi) = \xi(\cdot \cap S_r)$ and $\pi_r^c(\xi) = \xi(\cdot \cap S_r^c)$. For two measures ν_1, ν_2 on a measurable space $(\Omega, \mathcal{F})$ we write $\nu_1 \le \nu_2$ if $\nu_1(A) \le \nu_2(A)$ for any $A \in \mathcal{F}$. We can now state the definition of a *quasi-Gibbs measure* [13, 14].

Definition 2.1. A probability measure μ on $\mathfrak{M}$ is said to be a (Φ, Ψ)-quasi Gibbs measure if its regular conditional probabilities

$$
\mu_{r,\xi}^m(d\zeta) = \mu(d\zeta | \pi_r^c(\zeta) = \pi_r^c(\xi), \zeta(S_r) = m), \quad r, m \in \mathbb{N},
$$

satisfy that, for μ-a.s. ξ,

$$
c^{-1} e^{-H_r(\eta)} \Lambda_r^m(\pi_{S_r} \in d\eta) \le \mu_{r,\xi}^m(\pi_{S_r} \in d\eta) \le c e^{-H_r(\eta)} \Lambda_r^m(\pi_{S_r} \in d\eta).
$$

Here, $c = c(r, m, \xi)$ is a positive constant depending on r, m, and ξ.

It is readily seen that the quasi-Gibbs property is a generalized notion of the canonical Gibbs property. If μ is a (Φ, Ψ)-quasi Gibbs measure, then μ is also a $(\Phi + \Phi_{\mathrm{loc.bdd}}, \Psi)$-quasi Gibbs measure for any locally bounded measurable function $\Phi_{\mathrm{loc.bdd}}$. In this sense, the notion of "quasi-Gibbs" seems to be robust. Information about the free potential of μ is determined from its *logarithmic derivative* [12].

A function f on $\mathfrak{M}$ is called a polynomial function if

$$f(\xi) = Q\left(\langle \phi_1, \xi\rangle, \langle \phi_2, \xi\rangle, \dots, \langle \phi_\ell, \xi\rangle\right) \tag{2.4}$$

with $\phi_k \in C_c^\infty(S)$ and a polynomial function Q on $\mathbb{R}^\ell$, where $\langle \phi, \xi\rangle = \int_S \phi(x)\xi(dx)$ and $C_c^\infty(S)$ is the set of smooth functions with compact support. We denote by $\mathcal{P}$ the set of all polynomial functions on $\mathfrak{M}$.

Definition 2.2. We call $\mathbf{d}^\mu \in L^1_{loc}(S\times\mathfrak{M}, \mu^{[1]})$ the logarithmic derivative of μ if

$$\int_{S\times\mathfrak{M}} \mathbf{d}^\mu(x,\eta) f(x,\eta) d\mu^{[1]}(x,\eta) = -\int_{S\times\mathfrak{M}} \nabla_x f(x,\eta) d\mu^{[1]}(x,\eta)$$

is satisfied for $f \in C_c^\infty(S)\otimes\mathcal{P}$. Here $\mu^{[k]}$ is the Campbell measure of μ

$$\mu^{[k]}(A\times B) = \int_A \mu_{\mathbf{x}}(B)\rho^k(\mathbf{x})d\mathbf{x}, \quad A\in\mathcal{B}(S^k), B\in\mathcal{B}(\mathfrak{M}),$$

$\mu_{\mathbf{x}}$ is the reduced Palm measure conditioned at $\mathbf{x}\in S^k$

$$\mu_{\mathbf{x}} = \mu\left(\cdot - \sum_{j=1}^k \delta_{x_j} \middle| \xi(x_j) \ge 1 \text{ for } j=1,2,\dots,k\right), \tag{2.5}$$

and ρ^k is the k-correlation function for $k\in\mathbb{N}$.

Quasi-Gibbs measures inherit the following property from canonical Gibbs measures [19, Lemma 11.2]. Let $\mathcal{T}(\mathfrak{M})$ be the tail σ-field

$$\mathcal{T}(\mathfrak{M}) = \bigcap_{r=1}^\infty \sigma(\pi_r^c)$$

and let μ^ξ_{Tail} be the regular conditional probability defined as

$$\mu^\xi_{\mathrm{Tail}} = \mu(\cdot|\mathcal{T}(\mathfrak{M}))(\xi). \tag{2.6}$$

Then the following decomposition holds:

$$\mu(\cdot) = \int_{\mathfrak{M}} \mu^\xi_{\mathrm{Tail}}(\cdot)\mu(d\xi). \tag{2.7}$$

Furthermore, there exists a subset $\mathfrak{M}_0$ of $\mathfrak{M}$ satisfying $\mu(\mathfrak{M}_0)=1$ and, for all $\xi,\eta\in\mathfrak{M}_0$:

$$\mu^\xi_{\mathrm{Tail}}(A)\in\{0,1\} \quad \text{for all } A\in\mathcal{T}(\mathfrak{M}), \tag{2.8}$$

$$\mu^\xi_{\mathrm{Tail}}(\{\zeta\in\mathfrak{M} : \mu^\xi_{\mathrm{Tail}} = \mu^\zeta_{\mathrm{Tail}}\}) = 1, \tag{2.9}$$

$$\mu^\xi_{\mathrm{Tail}} \text{ and } \mu^\eta_{\mathrm{Tail}} \text{ are mutually singular on } \mathcal{T}(\mathfrak{M}) \text{ if } \mu^\xi_{\mathrm{Tail}} \ne \mu^\eta_{\mathrm{Tail}}. \tag{2.10}$$

§ 3. General theory of solutions of ISDEs

A polynomial function f on $\mathfrak{M}$ is a *local* function, that is, a function satisfying $f(\xi) = f(\pi_r(\xi))$ for some $r \in \mathbb{N}$. When $\xi \in \mathfrak{M}_r^m$, $m \in \mathbb{N} \cup \{0\}$ and $\pi_r(\xi)$ is represented by $\sum_{j=1}^m \delta_{x_j}$, we can regard $f(\xi) = f(\sum_{j=1}^m \delta_{x_j})$ as a permutation invariant smooth function on S_r^m. For $f, g \in \mathcal{P}$, define

$$\mathbb{D}(f,g)(\xi) = \frac{1}{2}\sum_{j=1}^{\infty} \nabla_{x_j} f(\xi) \cdot \nabla_{x_j} g(\xi).$$

For a probability μ on $\mathfrak{M}$, we denote by $L^2(\mathfrak{M}, \mu)$ the space of square integrable functions on $\mathfrak{M}$ with the inner product $\langle \cdot, \cdot \rangle_\mu$ and the norm $\|\cdot\|_{L^2(\mathfrak{M},\mu)}$. We consider the bilinear form $(\mathcal{E}^\mu, \mathcal{P}^\mu)$ on $L^2(\mathfrak{M}, \mu)$ defined by

$$\mathcal{E}^\mu(f,g) = \int_{\mathfrak{M}} \mathbb{D}(f,g) d\mu, \quad \mathcal{P}^\mu = \{f \in \mathcal{P} : \|f\|_1^2 < \infty\}, \tag{3.1}$$

where $\|f\|_1^2 \equiv \mathcal{E}^\mu(f,f) + \|f\|^2_{L^2(\mathfrak{M},\mu)}$.

We make the following assumptions

(A.0) μ has a locally bounded n-correlation function ρ^n for each $n \in \mathbb{N}$.

(A.1) There exist upper semi-continuous functions $\Phi_0 : S \to \mathbb{R} \cup \{\infty\}$ and $\Psi_0 : S \times S \to \mathbb{R} \cup \{\infty\}$ that are locally bounded from below, and $c > 0$ such that

$$c^{-1}\Phi_0(x) \leq \Phi(x) \leq c\Phi_0(x), \quad c^{-1}\Psi_0(x,y) \leq \Psi(x,y) \leq c\Psi_0(x,y).$$

(A.2) There exists a $T > 0$ such that for each $R > 0$

$$\liminf_{r\to\infty} \mathrm{Erf}\left(\frac{r}{(r+R)T}\right) \int_{|x|\leq r+R} \rho^1(x)dx = 0,$$

where $\mathrm{Erf}(t) = (2\pi)^{-1/2}\int_t^\infty e^{-x^2/2}dx$.

Note that $\mathcal{P}^\mu = \mathcal{P}$ and $(\mathcal{E}^\mu, \mathcal{P}^\mu) = (\mathcal{E}, \mathcal{P})$ under condition (A.0).

Theorem 3.1 ([12, 13, 14, 11, 16])**.** *Suppose that μ is a (Φ, Ψ)-quasi Gibbs measure satisfying (A.0) and (A.1). Then*
(i) $(\mathcal{E}, \mathcal{P})$ is closable and its closure $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is a quasi regular Dirichlet form and there exists the diffusion process $(\Xi(t), P_\mu^\xi)$ associated with $(\mathcal{E}^\mu, \mathcal{D}^\mu)$.
(ii) Furthermore, assume conditions (A.2) and (A.3):
(A.3) $\mathrm{Cap}^\mu((\mathfrak{M}_{\mathrm{s.i}})^c) = 0$ *and* $\mathrm{Cap}^\mu(\xi(\partial S) \geq 1) = 0$,
where Cap^μ is the capacity of the Dirichlet form. If there exists a logarithmic derivative $\mathbf{d}^\mu$, then there exists $\tilde{\mathfrak{M}} \subset \mathfrak{M}$ such that $\mu(\tilde{\mathfrak{M}}) = 1$, and for any $\xi = \sum_{j\in\mathbb{N}} \delta_{x_j} \in \tilde{\mathfrak{M}}$, there exists an $S^{\mathbb{N}}$-valued continuous process $\mathbf{X}(t) = (X_j(t))_{j=1}^\infty$ satisfying $\mathbf{X}(0) = \mathbf{x} = (x_j)_{j=1}^\infty$ and

$$dX_j(t) = dB_j(t) + \frac{1}{2}\mathbf{d}^\mu\Big(X_j(t), \sum_{k:k\neq j} \delta_{X_k(t)}\Big)dt, \quad j \in \mathbb{N}.$$

Let $\mathfrak{l}$ be a label map from $\mathfrak{M}_{\mathrm{s.i.}}$ to $S^{\mathbb{N}}$, that is, for each $\xi \in \mathfrak{M}_{\mathrm{s.i.}}$, $\mathfrak{l}(\xi) = (\mathfrak{l}(\xi)_j)_{j=1}^{\infty} \in S^{\mathbb{N}}$ satisfies $\xi = \sum_{j=1}^{\infty} \delta_{\mathfrak{l}(\xi)_j}$. The map $\mathfrak{l}$ can be lifted to the map from $C([0,\infty), \mathfrak{M}_{s.i.})$ to $C([0,\infty), S^{\mathbb{N}})$. For $\Xi \in C([0,\infty), \mathfrak{M}_{\mathrm{s.i.}})$ we put

$$\Xi^{\diamond m}(t) = \sum_{j=m+1}^{\infty} \delta_{X_j(t)}$$

for each $m \in \mathbb{N}$, where $(X_j)_{j=1}^{\infty} = \mathfrak{l}(\Xi) \in C([0,\infty), S^{\mathbb{N}})$. We make the following assumption.

(A4) There exists a subset $\mathfrak{M}_{\mathrm{SDE}}$ of $\mathfrak{M}_{s.i.}$ such that

$$P_{\mu}^{\xi}(\Xi(t) \in \mathfrak{M}_{\mathrm{SDE}}) = 1 \quad \text{for any } \xi \in \mathfrak{M}_{\mathrm{SDE}},$$

and for each $\Xi \in C([0,\infty), \mathfrak{M}_{\mathrm{SDE}})$ and each $m \in \mathbb{N}$,

$$\begin{aligned} \text{(3.2)} \quad dY_j^{(m)}(t) = dB_j(t) - \frac{1}{2}\nabla\Phi(Y_j^{(m)}(t))dt - \frac{1}{2}\sum_{k=1,k\neq j}^{m} \nabla\Psi(Y_j^{(m)}(t), Y_k^{(m)}(t))dt \\ -\frac{1}{2}\int_{\mathfrak{M}} \nabla\Psi(Y_j^{(m)}(t), X(t))\Xi^{\diamond m}(dX)dt, \quad 1 \le j \le m, \end{aligned}$$

$$\text{(3.3)} \quad Y_j^{(m)}(0) = \mathfrak{l}(\Xi(0))_j, \quad 1 \le j \le m,$$

has a unique strong solution $\mathbf{Y}^{(m)} = (Y_1^{(m)}, Y_2^{(m)}, \ldots, Y_m^{(m)})$.

We also make the following assumptions about the probability measure μ

(A5) For each $r, T \in \mathbb{N}$, there exists a positive constant c such that

$$\int_S \mathrm{Erf}\left(\frac{|x| - r}{\sqrt{cT}}\right) \rho^1(x)dx < \infty.$$

(A6) The tail σ-field $\mathcal{T}(\mathfrak{M})$ is μ-trivial, that is, $\mu(A) \in \{0,1\}$ for $A \in \mathcal{T}(\mathfrak{M})$.

Definition 3.2. Let μ be a probability measure on $\mathfrak{M}$ and let $\Xi(t)$ be an $\mathfrak{M}$-valued process. We say that $\Xi(t)$ satisfies the *μ-absolute continuity condition* if $\mu \circ \Xi(t)^{-1}$ is absolutely continuous with respect to μ for $\forall t > 0$. We say that an $S^{\mathbb{N}}$-valued process $\mathbf{X}(t)$ satisfies the μ-absolute continuity condition if $\mathfrak{u}(\mathbf{X}(t))$ satisfies the μ-absolute continuity condition, where $\mathfrak{u}$ is the map from $S^{\mathbb{N}}$ to $\mathfrak{M}$ defined by $\mathfrak{u}((x_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \delta_{x_j}$.

Then we have the following theorem.

Theorem 3.3 ([19])**.** *Suppose that the assumptions in Theorem 3.1 are satisfied. Furthermore assume (A4)–(A6). Then, for μ-a.s. ξ, ISDE (1.6) with $\mathbf{X}(0) = \mathfrak{l}(\xi)$ has a strong solution satisfying the μ-absolute continuity condition, and that pathwise uniqueness holds for ISDE (1.6) with the μ-absolute continuity condition.*

§ 4. Applications

Theorems 3.1 and 3.3 can be applied to quite general class of ISDEs. In this section we give some important examples.

Example 4.1 (Canonical Gibbs measures). Let $S = \mathbb{R}^d$, $d \in \mathbb{N}$. Assume that $\Phi = 0$ and that Ψ_0 is a super stable and regular in the sense of Ruelle [22], and is smooth outside the origin. Let μ be a canonical Gibbs measure with the interaction Ψ_0. Then its logarithmic derivative is

$$\mathbf{d}^{\mu}\Big(x, \sum_{k:k\neq j} \delta_{y_k}\Big) = -\sum_{k=1,k\neq j}^{\infty} \nabla\Psi_0(x-y_k). \tag{4.1}$$

Assume that (A.2) is satisfied. In the case $d \geq 2$, there exists a diffusion process associated with μ and the labeled process solves

$$dX_j(t) = dB_j(t) - \frac{1}{2}\sum_{k=1,k\neq j}^{\infty} \nabla\Psi_0(X_j(t)-X_k(t))dt. \tag{4.2}$$

In the case $d = 1$, Ψ_0 needs to be sufficient repulsive at the origin to satisfy (A.3).

Assume that (A.5) is satisfied and that, for each $n \in \mathbb{N}$, there exist positive constants c, c' satisfying

$$\sum_{r=1}^{\infty} \frac{\int_{|x|>r} \rho^1(x)dx}{r^c} < \infty, \tag{4.3}$$

$$\sum_{i,j=1}^{d} \left|\frac{\partial^2}{\partial x_i \partial x_j}\Psi_0(x)\right| \leq \frac{c'}{(1+|x|)^{c'+1}}, \tag{4.4}$$

for all $|x| \geq 1/n$. In [19, Theorem 3.3] it was proved that, for μ-a.s. ξ, ISDE (4.2) with $\mathbf{X}(0) = \mathfrak{l}(\xi)$ has a strong solution satisfying the $\mu_{\mathrm{Tail}}^{\xi}$-absolute continuity condition, and that pathwise uniqueness holds for ISDE (1.6) with the $\mu_{\mathrm{Tail}}^{\xi}$-absolute continuity condition.

Example 4.2 (Sine random point fields). Let $\check{\mu}_{\mathsf{bulk},\beta}^{N}$ be the probability measure defined in (1.3). We denote by $\mu_{\mathsf{bulk},\beta}^{N}$ the distribution of $\sum_{j=1}^{N} \delta_{x_j}$ under $\check{\mu}_{\mathsf{bulk},\beta}^{N}$. For $\beta > 0$ the existence of the limit of $\mu_{\mathsf{bulk},\beta}^{N}$ as $N \to \infty$ was shown in Valkó-Virág[24]. We denote the limit by $\mu_{\mathsf{bulk},\beta}$. In particular, when $\beta = 2$, $\mu_{\mathsf{bulk},2}$ is the determinantal point process (DPP) with the sine kernel

$$K_{\sin,2}(x,y) = \frac{\sin(x-y)}{\pi(x-y)}, \tag{4.5}$$

and when $\beta = 1, 4$, it is a quaternion determinantal point process [2]. It was shown that $\mu_{\mathsf{bulk},\beta}$ for $\beta = 1, 2, 4$ is a quasi-Gibbs measure in [13], and that its logarithmic derivative is

$$\mathbf{d}^{\mu}\Big(x, \sum_{k:k\neq j} \delta_{y_k}\Big) = \beta \lim_{r\to\infty} \sum_{\substack{k:k\neq j \\ |y_k|<r}} \frac{1}{x-y_k} \tag{4.6}$$

in [12]. In [19, Theorem 3.1] it was shown that for $\mu_{\mathsf{bulk},\beta}$-a.s. ξ, ISDE (1.4) with $\mathbf{X}(0) = \mathfrak{l}(\xi)$ has a strong solution satisfying the $\mu^{\xi}_{\mathsf{bulk},\beta,\mathrm{Tail}}$-absolute continuity condition, and that pathwise uniqueness holds for ISDE (1.4) with the $\mu^{\xi}_{\mathsf{bulk},\beta,\mathrm{Tail}}$-absolute continuity condition. In the case $\beta = 2$, the facts that $\mathcal{T}(\mathfrak{M})$ is $\mu_{\mathsf{bulk},2}$-trivial and $\mu^{\xi}_{\mathsf{bulk},2,\mathrm{Tail}} = \mu_{\mathsf{bulk},2}$ were shown in [15].

Tsai [23] proved the existence and uniqueness of solutions of ISDE (1.4) for $\beta \geq 1$ by a different method. Thus it is conjectured that $\mu_{\mathsf{bulk},\beta}$ is a quasi-Gibbs measure and has a logarithmic derivative of the form (4.6) for $\beta \geq 1$.

Example 4.3 (Airy random point fields). We denote by $\mu^{N}_{\mathsf{soft},\beta}$ the distribution of $\sum_{j=1}^{N} \delta_{N^{1/6}(x_j-2\sqrt{N})}$ under $\check{\mu}^{N}_{\mathsf{bulk},\beta}$. For $\beta > 0$, the existence of the limit of $\mu^{N}_{\mathsf{soft},\beta}$ as $N \to \infty$ was shown in Ramírez-Rider-Virág [21]. We denote the limit by $\mu_{\mathsf{soft},\beta}$. In particular, when $\beta = 2$, $\mu_{\mathsf{soft},2}$ is the DPP with the Airy kernel

$$K_{\mathrm{Ai},2}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, \tag{4.7}$$

where Ai denotes the Airy function and Ai$'$ its derivative [9]. When $\beta = 1, 4$, it is a quaternion determinantal point process [2]. In the cases $\beta = 1, 2, 4$, it has been proved that the random point field is quasi-Gibbsian [14], and that its logarithmic derivative is

$$\mathbf{d}^{\mu}\Big(x, \sum_{k:k\neq j} \delta_{y_k}\Big) = \beta \lim_{r\to\infty} \left\{ \sum_{\substack{k:k\neq j \\ |y_k|<r}} \frac{1}{x-y_k} - \int_{-r}^{r} \frac{\widehat{\rho}(x)dx}{-x} \right\}, \tag{4.8}$$

and for $\mu_{\mathsf{soft},\beta}$-a.s. ξ, ISDE (1.5) with $\mathbf{X}(0) = \mathfrak{l}(\xi)$ has a strong solution satisfying the $\mu^{\xi}_{\mathsf{soft},\beta,\mathrm{Tail}}$-absolute continuity condition, and pathwise uniqueness holds for ISDE (1.5) with the $\mu^{\xi}_{\mathsf{soft},\beta,\mathrm{Tail}}$-absolute continuity condition [18, Theorem 2.3]. In the case $\beta = 2$ the facts that $\mathcal{T}(\mathfrak{M})$ is $\mu_{\mathsf{soft},2}$-trivial and that $\mu^{\xi}_{\mathsf{soft},2,\mathrm{Tail}} = \mu_{\mathsf{soft},2}$ were shown in [15].

Determining whether $\mu_{\mathsf{soft},\beta}$ has the quasi-Gibbs property for general β and finding its logarithmic derivative is (4.8) are interesting and important problems.

Example 4.4 (Bessel random point field). Let $S = [0, \infty)$ and $1 \leq \alpha < \infty$. Let $\mu_{\mathsf{hard},2}$ be the determinantal point process with Bessel kernel

$$K_{J_\alpha}(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}. \tag{4.9}$$

In [6] it was shown that $\mu_{\mathsf{hard},2}$ is a quasi-Gibbs measure and that the related process is the unique strong solution of the ISDE

$$dX_j(t) = dB_j(t) + \left\{ \frac{\alpha}{2X_j(t)} + \sum_{k=1,k\neq j}^{\infty} \frac{1}{X_j(t)-X_k(t)} \right\} dt$$

with the $\mu_{\mathsf{hard},2}$-absolute continuity condition.

Example 4.5 (Ginibre random point field)**.** Let $S=\mathbb{R}^2$ be identified as $\mathbb{C}$. Let μ_{Gin} be the DPP with the kernel $K_{\mathsf{Gin}} : \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ defined by

$$K_{\mathsf{Gin}}(x,y) = \frac{1}{\pi} e^{-|x|^2/2-|y|^2/2} e^{x\overline{y}}. \tag{4.10}$$

In [13] it was shown that μ_{Gin} is a quasi-Gibbs measure, and in [12] that the related process is a solution of the ISDE

$$dX_j(t) = dB_j(t) - X_j(t)dt + \lim_{r\to\infty} \sum_{\substack{k:k\neq j \\ |X_k(t)|<r}} \frac{X_j(t)-X_k(t)}{|X_j(t)-X_k(t)|^2} dt. \tag{4.11}$$

The pathwise uniqueness of solutions of (4.11) with the μ_{Gin}-absolute continuity condition was shown in [19].

§ 5. Remarks

In the previous section we gave some examples of DPPs that are not canonical Gibbs measures but quasi-Gibbs measures. It is expected that quite general DPPs have the quasi-Gibbs property. We thus present examples of DPPs related to random matrix theory or non-colliding Brownian motions, whose quasi-Gibbs property have not been shown.

Example 5.1 (Pearcey process)**.** Consider $2N$ noncolliding Brownian motions, in which all particles start from the origin and N particles end at $\sqrt{N}$ at time $t=1$, and the other N particles end at $-\sqrt{N}$ at $t=1$. We denote the system by $(X_1^N(t),\dots,X_{2N}^N(t))$, $0\le t\le 1$. When N is very large, there is a cusp at $x_0^N=0$ when $t_0=\frac{1}{2}$, that is, before time t_0 particles are in one interval with high probability, while after time t_0 they are separated into two intervals by the origin. We denote the distribution

$$\sum_{j=1}^{2N} \delta_{2^{3/2}(2N)^{1/4}X_j^N(\frac{1}{2})}$$

on $\mathfrak{M}$ by μ^N_{pearcey}. It was proved in Adler-Orantin-von Moerbeke [1] that

$$\mu^N_{\mathsf{pearcey}} \to \mu_{\mathsf{pearcey}}, \quad \text{weakly as } N \to \infty$$

and that μ_{pearcey} is the DPP $K_{\mathsf{pearcey}}(x,y)$ given by

$$K_{\mathsf{pearcey}}(x,y) = \frac{P(x)Q''(y) - P'(x)Q'(y) + P''(x)Q(y)}{x-y}, \quad x, y \in \mathbb{R},$$

with

$$Q(y) = \frac{i}{2\pi}\int_{-i\infty}^{i\infty} e^{-u^4/4-uy}du \quad \text{and} \quad P(x) = \frac{1}{2\pi i}\int_C e^{v^4/4+vx}dv,$$

where the contour C is given by the ingoing rays from $\pm\infty e^{i\pi/4}$ to 0 and the outgoing rays from 0 to $\pm\infty e^{-i\pi/4}$. These integrals are known as Pearcey's integrals [20].

Example 5.2 (Tacnode process)**.** Consider two groups of non-colliding pinned Brownian motions $(X^N_1(t), \dots, X^N_{2N}(t))$ in the time interval $0 \le t \le 1$, where one group of N particles starts and ends at $\sqrt{N}$ and the other group of N particles starts and ends at $-\sqrt{N}$. The distribution $(N^{1/6}X^N_1(\frac{1}{2}), N^{1/6}X^N_2(\frac{1}{2}), \dots, N^{1/6}X^N_{2N}(\frac{1}{2}))$ on the Weyl chamber of type A_{2N-1}

$$\mathbb{W}_{2N} = \Big\{\mathbf{x} = (x_1, x_2, \cdots, x_{2N}) : x_1 < x_2 < \cdots < x_{2N}\Big\},$$

is given by

$$m^{2N}_{\mathsf{tac}}(d\mathbf{x}_{2N}) = \frac{1}{Z}\left[\det_{1\le i,j\le 2N}\left(e^{-2|x_i-a_j|^2}\right)\right]^2,$$

where $a_j = -\sqrt{N}$ for $1 \le j \le N$ and $a_j = \sqrt{N}$ for $N+1 \le j \le 2N$. We denote the distribution of $\sum_{j=1}^{2N}\delta_{N^{1/6}x_j}$ under m^{2N}_{tac} by μ^N_{tac}. It was proved in Delvaux-Kuijlaars-Zhang [3] and Johansson [7] that

$$\mu^N_{\mathsf{tac}} \to \mu_{\mathsf{tac}}, \quad \text{weakly as } N \to \infty$$

and that μ_{tac} is the DPP with the correlation kernel

$$K_{\mathsf{tac}}(x,y) \equiv L_{\mathsf{tac}}(x,y) + L_{\mathsf{tac}}(-x,-y), \quad x, y \in \mathbb{R},$$

where

$$\begin{aligned}
L_{\mathsf{tac}}(x,y) &= K_{\mathrm{Ai},2}(x,y) \\
&\quad + 2^{1/3}\int_{(0,\infty)^2} dudv\ \mathrm{Ai}(y + 2^{1/3}u)R(u,v)\mathrm{Ai}(x + 2^{1/3}v) \\
&\quad - 2^{1/3}\int_{(0,\infty)^2} dudv\ \mathrm{Ai}(-y + 2^{1/3}u)\mathrm{Ai}(u+v)\mathrm{Ai}(x + 2^{1/3}v) \\
&- 2^{1/3}\int_{(0,\infty)^3} dudvdw\ \mathrm{Ai}(-y + 2^{1/3}u)R(u,v)\mathrm{Ai}(v+w)\mathrm{Ai}(x + 2^{1/3}w).
\end{aligned}$$

Here, $R(x, y)$ is the resolvent operator for the restriction of the Airy kernel to $[0, \infty)$, that is, the kernel of the operator

$$R = (I - K_{\mathrm{Ai}})^{-1} K_{\mathrm{Ai}} \tag{5.1}$$

on $L^2[0, \infty)$.

In [3, 7] it was also shown that

$$\Xi^N(t) \equiv \sum_{j=1}^{2N} \delta_{N^{1/6} X_j(\frac{1}{2} + N^{-1/3} t)} \to \Xi(t), \qquad \text{as } N \to \infty,$$

in the sense of finite-dimensional distributions, where $\Xi(t)$ is a reversible process with reversible measure μ_{tac}. We expect that $\Xi(t)$ is the diffusion process associated with the Dirichlet form $(\mathcal{E}^{\mu_{\mathsf{tac}}}, \mathcal{D}^{\mu_{\mathsf{tac}}})$.

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